

Small connections are cyclic

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ABSTRACT

The main local invariants of a (one variable) differential module over the complex numbers are given by means of a cyclic basis. In the p -adic setting the existence of a cyclic vector is often unknown. We investigate the existence of such a cyclic vector in a Banach algebra. We follow the explicit method of Katz [Kat87], and we prove the existence of such a cyclic vector under the assumption that the matrix of the derivation is small enough in norm.

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1. Katz's simple algorithm for cyclic vectors

Let (\mathcal{B}, d) be a commutative ring \mathcal{B} with unit, together with a derivation¹ $d : \mathcal{B} \rightarrow \mathcal{B}$. We denote by $\mathcal{B}^{d=0} := \{b \in \mathcal{B} \text{ such that } d(b) = 0\}$ the sub-ring of constants. A differential module M is a free \mathcal{B} -module of finite rank together with an action of the derivation

$$\nabla : M \rightarrow M \tag{1.1}$$

i.e. a \mathbb{Z} -linear map satisfying $\nabla(bm) = d(b)m + b\nabla(m)$ for all $b \in \mathcal{B}$, $m \in M$. A cyclic vector for M is an element $m \in M$ such that the family $\{m, \nabla(m), \nabla^2(m), \dots, \nabla^{n-1}(m)\}$ is a basis of M over \mathcal{B} . Such a vector does not always exists. Namely if $d = 0$ is the trivial derivation, then ∇ is merely a \mathcal{B} -linear map and (M, ∇) is a torsion module over the ring of polynomials $\mathcal{B}[X]$ where the action of X on M is given by ∇ . There is another counterexample in the case in which $\mathcal{B} = \mathbb{F}_p(X)$ is a functions field in characteristic $p > 0$: let $M := \mathbb{F}_q[X]^n$, with $n > q = p^r$, together with the trivial connection $\nabla(f_1, \dots, f_n) = (f'_1, \dots, f'_n)$, then, since $d^q = 0$, one has $\nabla^q = 0$ so M does not have any cyclic vector. The same happens replacing \mathbb{F}_q by a ring A having a maximal ideal \mathfrak{m} such that $A/\mathfrak{m} \cong \mathbb{F}_q$. The trivial connection of $A[X]^n$ (with respect to d/dx) can not admit a cyclic vector, since otherwise its reduction to $\mathbb{F}_q[X]$ would be cyclic too.

2000 Mathematics Subject Classification ????

Keywords: cyclic vector, p -adic differential equations

¹i.e. a \mathbb{Z} -linear map satisfying the Leibnitz rule $d(ab) = ad(b) + d(a)b$

1.1 Three cyclic vector theorems.

P.Deligne provided the existence of such a cyclic vector for all differential modules over a field of characteristic 0 with non trivial derivation (cf. [Del70, Ch.II,Lemme 1.3]).

THEOREM 1.1 [Del70, Ch.II, Lemme 1.3]. *Let \mathcal{B} be a field of characteristic 0, then all differential modules over \mathcal{B} admit a cyclic vector.*

Subsequently N.Katz generalized the result of Deligne providing the following simple explicit algorithm:

THEOREM 1.2 ([Kat87]). *Assume that there exists an element $t \in \mathcal{B}$ such that $d(t) = 1$. Assume moreover that $(n-1)!$ is invertible in \mathcal{B} , and that $\mathcal{B}^{d=0}$ contains a field k such that $^2 \#k > n(n-1)$. Let $a_0, a_1, \dots, a_{n(n-1)}$ be $n(n-1) + 1$ distinct elements of k , and let $\mathbf{e} := \{e_0, \dots, e_{n-1}\} \subset M$ be a basis of M over \mathcal{B} . Then Zarisky locally on $\text{Spec}(\mathcal{B})$ one of the vectors*

$$c(\mathbf{e}, t - a_i) := \sum_{j=0}^{n-1} \frac{(t - a_i)^j}{j!} \sum_{k=0}^j (-1)^k \binom{j}{k} \nabla^k(e_{j-k}) \quad (1.2)$$

is a cyclic vector of M .

THEOREM 1.3 ([Kat87]). *If \mathcal{B} is a local $\mathbb{Z}[1/(n-1)!]$ -algebra, and if $a \in \mathcal{B}^{d=0}$ is such that the maximal ideal of \mathcal{B} contains $t - a$, then $c(\mathbf{e}, t - a)$ is a cyclic vector for M .*

The arguments of the Katz's proofs are the following. We consider the polynomial ring $\mathcal{B}[X]$ and we extend the derivation of \mathcal{B} by $d(X) = 1$. We denote again by ∇ the action of d on $M \otimes_{\mathcal{B}} \mathcal{B}[X]$ given by $\nabla \otimes \text{Id}_{\mathcal{B}[X]} + \text{Id}_M \otimes d$. Each element c_0 in $M \otimes_{\mathcal{B}} \mathcal{B}[X]$ can be uniquely represented as $c_0 := \sum_{j \geq 0} c_{0,j} X^j$, with $c_{0,j} \in M$ for $j = 0, 1, \dots$. The derivatives $\nabla^i(c_0)$ of c_0 then have the same form $c_i := \nabla^i(c_0) = \sum_{j \geq 0} c_{i,j} X^j$, with $c_{i,j} = \sum_{k=0}^i k! \binom{j+k}{j} \binom{i}{k} \nabla^{i-k}(c_{0,j+k})$.

$$\begin{array}{rclclclclcl} c_0 & = & c_{0,0} & + & c_{0,1} \cdot X & + & c_{0,2} \cdot X^2 & + & \dots & + & c_{0,n-1} \cdot X^{n-1} & + & \dots \\ \nabla(c_0) & = & c_{1,0} & + & c_{1,1} \cdot X & + & c_{1,2} \cdot X^2 & + & \dots & + & c_{1,n-1} \cdot X^{n-1} & + & \dots \\ \nabla^2(c_0) & = & c_{2,0} & + & c_{2,1} \cdot X & + & c_{2,2} \cdot X^2 & + & \dots & + & c_{2,n-1} \cdot X^{n-1} & + & \dots \\ \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots \\ \nabla^{n-1}(c_0) & = & c_{n-1,0} & + & c_{n-1,1} \cdot X & + & c_{n-1,2} \cdot X^2 & + & \dots & + & c_{n-1,n-1} \cdot X^{n-1} & + & \dots \\ \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots \end{array} \quad (1.3)$$

The main point is now that, if $(n-1)!$ is invertible in \mathcal{B} , and if the degree (with respect to X) of c_0 is less or equal to $n-1$, then the 0-components $\{c_{0,0}, c_{1,0}, \dots, c_{n-1,0}\}$ of $\{c_0, \nabla(c_0), \dots, \nabla^{n-1}(c_0)\}$ uniquely determine c_0 . In fact we have the inversion formula

$$c_{0,j} := \frac{1}{j!} \sum_{k=0}^j (-1)^{j-k} \binom{j}{k} \nabla^{j-k}(c_{k,0}), \quad j = 0, \dots, n-1. \quad (1.4)$$

The idea is then to choose the 0-components equal to the basis of M : $c_{k,0} := e_k$. We then obtain the vector (1.2):

$$c(\mathbf{e}, X) := \sum_{j=0}^{n-1} \frac{X^j}{j!} \sum_{k=0}^j (-1)^k \binom{j}{k} \nabla^k(e_{j-k}). \quad (1.5)$$

This choice implies that the determinant of the base change is a polynomial $P(X) \in \mathcal{B}[X]$ verifying $P(0) = 1$, because the matrix $H(X) \in M_n(\mathcal{B}[X])$ expressing $\{c_0, \nabla(c_0), \dots, \nabla^{n-1}(c_0)\}$ in the basis \mathbf{e} verifies $H(0) = \text{Id}$. In other words $P(X)$ is invertible as a formal power series in $\mathcal{B}[[X]]$, so that c_0 is a cyclic vector for $M \otimes_{\mathcal{B}} \mathcal{B}[[X]]$.

²The symbol $\#k$ means the number of elements of k , or, if k is infinite, its cardinality.

We now specialize X into an element $t - a$ verifying $d(t - a) = 1$, this guarantee that the specialization commutes with the action of the derivation. Let us come to the proof of the above results. If \mathcal{B} is local, and if $t - a$ belongs to the maximal ideal, then $P(t - a)$ is clearly invertible since it is of the form $P(t - a) = P(0) + (t - a)Q(t - a) = 1 + y$, with y in the maximal ideal. This proves theorem 1.3. Notice that if \mathcal{B} is a field of characteristic 0, then $\mathcal{B}^{d=0}$ is an infinite field, hence there exists at least a constant $a \in \mathcal{B}^{d=0}$ such that $P(t - a) \neq 0$, this is enough to prove Deligne's Theorem 1.1.³ Now we come to the proof of Theorem 1.2. Katz proves that the ideal \mathcal{I} of \mathcal{B} generated by the values $\{P(t - a_i)\}_{i=0, \dots, n(n-1)}$ is the unit ideal. He argues as follows. We observe that the polynomial $P(X)$ has degree $\leq n(n-1)$, since

$$c_0 \wedge \nabla(c_0) \wedge \dots \wedge \nabla^{n-1}(c_0) = P(X) \cdot e_0 \wedge e_1 \wedge \dots \wedge e_{n-1} \quad (1.6)$$

and the n vectors $c_0, \nabla(c_0), \dots, \nabla^{n-1}(c_0)$ have all degree $\leq (n-1)$. So we write $P = \sum_{s=0}^{n(n-1)} r_s X^s$ and $P(t - a_i) = \sum_{s=0}^{n(n-1)} r_s (t - a_i)^s$. Now for $i \neq j$ one has $(t - a_i) - (t - a_j) = a_j - a_i \neq 0$ in k , so $(t - a_i) - (t - a_j)$ is invertible in \mathcal{B} . Hence the Van Der Monde matrix $V := ((t - a_i)^j)_{0 \leq i, j \leq n(n-1)}$ is invertible because its determinant is $\prod_{0 \leq i < j \leq n(n-1)} (a_j - a_i)$. This implies that the ideal \mathcal{I} is equal to the ideal generated by the coefficients $r_0, \dots, r_{n(n-1)}$.⁴ Since $r_0 = 1$, then $\mathcal{I} = \mathcal{B}$. This concludes the Katz's proofs.

1.1.1 About the assumptions of Katz's Theorems. The assumption about the existence of t such that $d(t) = 1$ is not completely constrictive. Indeed it is enough to assume the existence of an element $\tilde{t} \in \mathcal{B}$ such that $d(\tilde{t}) = f$ is invertible in \mathcal{B} . Then we replace the derivation d by $\tilde{d} := f^{-1} \cdot d$ in order to have $\tilde{d}(\tilde{t}) = 1$. We then consider the connection $\tilde{\nabla} := f^{-1} \cdot \nabla$ on M , and we form the Katz's cyclic vector (1.2) constructed from the data of $(\tilde{d}, \tilde{t}, \tilde{\nabla})$. Then

LEMMA 1.4. *The vector c is a cyclic vector for the differential module (M, ∇) over (\mathcal{B}, d) if and only if c is a cyclic vector for $(M, f \cdot \nabla)$ over $(\mathcal{B}, f \cdot d)$, for an arbitrary invertible element $f \in \mathcal{B}$.*

Proof. It is enough to prove that if c is cyclic with respect to (M, ∇) then it is a cyclic vector with respect to $(M, f \nabla)$. We have to prove that the base change matrix from the basis $\{c, \nabla(c), \dots, \nabla^{n-1}(c)\}$ to the family $\{c, (f \nabla)(c), (f \nabla)^2(c), \dots, (f \nabla)^{n-1}(c)\}$ is invertible. The Leibnitz rule of ∇ gives the relation $\nabla \circ f = f \circ \nabla + d(f)$ where f and $d(f)$ denote respectively the multiplication in M by $f \in \mathcal{B}$ and $d(f) \in \mathcal{B}$. One sees then that $(f \nabla)^k = f^k \nabla^k + \sum_{0 \leq i \leq k-1} \alpha_i(f) \nabla^i$, for convenient elements $\alpha_i(f) \in \mathcal{B}$. This implies that the base change matrix is triangular with $(1, f, f^2, \dots, f^{n-1})$ in the diagonal. \square

REMARK 1.5. *The Katz's algorithm is not invariant under the above change of derivation. In other words the Katz's vector c_0 obtained from (d, t, ∇) does not coincide with the Katz's vector \tilde{c}_0 constructed from $(\tilde{d}, \tilde{t}, \tilde{\nabla})$.⁵ If one of them is a cyclic vector, then it is simultaneously cyclic for ∇ and $\tilde{\nabla}$, thanks to the above lemma. But actually, in our knowledge, the fact that one of them is cyclic does not imply necessarily that the other is cyclic too.*

1.2 The Katz's base change matrix.

We now investigate the explicit form of the base change matrix $H(X)$. For this we need to introduce some notation. If a basis \mathbf{e} of M is fixed then we can associate to the n -times iterated connection

³Notice that Deligne does not ask for the existence of $t \in \mathcal{B}$ satisfying $d(t) = 1$. But it is easy to reduce the general case to this one by replacing the non trivial derivation d by $\tilde{d} := f \cdot d$, with $f := d(t)^{-1}$, and then using Lemma 1.4.

⁴The ideal \mathcal{I} is the set of linear combinations $\sum_{i=0}^{n(n-1)} b_i P(t - a_i) = {}^t w \cdot (P(t - a_i))_i$ with coefficients b_i in \mathcal{B} . Since V is invertible, for all vector v with coefficients in \mathcal{B} , there exists w such that ${}^t w \cdot V = {}^t v$ and reciprocally. So that any linear combination ${}^t w \cdot (P(t - a_i))_i$ of the family $\{P(t - a_i)\}_i$ is in fact a linear combination of the family $\{r_i\}_i$, because ${}^t w \cdot (P(t - a_i))_i = {}^t w \cdot V \cdot (r_i)_i = {}^t v \cdot (r_i)_i$, and reciprocally. So \mathcal{I} is the ideal generated by the family $\{r_i\}_i$.

⁵Notice that $d(t) = \tilde{d}(\tilde{t}) = 1$. Once we change d we also have to change t in order to preserve this relation.

$\nabla^n := \nabla \circ \nabla \circ \cdots \circ \nabla$ a matrix $G_n = (g_{n;i,j})_{i,j=0,\dots,n-1} \in M_n(\mathcal{B})$ whose rows are the image of the basis \mathbf{e} by ∇^n :

$$\nabla^n(e_i) := \sum_{j=0}^{n-1} g_{n;i,j} \cdot e_j . \quad (1.7)$$

PROPOSITION 1.6. *The Katz's base change matrix $H(X)$ verifying $(\nabla^i(c_0(\mathbf{e}, X)))_i = H(X)(e_i)_i$ has the form*

$$H(X) := H_0(X) + H_1(X) \cdot G_1 + \cdots + H_{2n-2}(X) \cdot G_{2n-2} , \quad (1.8)$$

where the matrices $H_s(X)$, $s = 0, \dots, 2n-2$, all belong to $\mathbb{Z}[\frac{1}{(n-1)!}][X]$ and satisfy the following properties:

i) One has

$$H_0(X) = \begin{pmatrix} 1 & X & \frac{X^2}{2} & \frac{X^3}{3!} & \cdots & \cdots & \frac{X^{n-1}}{(n-1)!} \\ 0 & 1 & X & \frac{X^2}{2} & \frac{X^3}{3!} & \cdots & \frac{X^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & X & \frac{X^2}{2} & \cdots & \frac{X^{n-3}}{(n-3)!} \\ 0 & 0 & 0 & 1 & X & \cdots & \frac{X^{n-4}}{(n-4)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 & X \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \quad (1.9)$$

ii) If $H_s(X) = (h_{s;i,j}(X))_{i,j}$ then

$$h_{s;i,j}(X) = \alpha(s; i, j) \frac{X^{s+j-i}}{(s+j-i)!} \quad (1.10)$$

with

$$\alpha(s; i, j) = \epsilon_{s;i,j} \cdot \left[\sum_{k=\max(0, s+j-(n-1))}^{\min(i, s)} (-1)^{s+k} \binom{s-k+j}{j} \binom{i}{k} \right] \in \mathbb{Z} \quad (1.11)$$

where

$$\epsilon_{s;i,j} = \begin{cases} 1 & \text{if } (s, j) \in [0, n-1+i] \times [\max(0, i-s), \min(n-1, n-1+i-s)] \\ 0 & \text{if } (s, j) \notin [0, n-1+i] \times [\max(0, i-s), \min(n-1, n-1+i-s)] \end{cases} \quad (1.12)$$

iii) In particular one has $h_{s;i,j} = 0$ if $j-i$ does not belong to the interval $[\max(1-s, 1-n), n-1-s]$.

Proof. Applying ∇^i to the vector $c_0(\mathbf{e}, X)$, and re-summing by setting $s := m + k - j$ one obtains

$$\nabla^i(c_0(\mathbf{e}, X)) = \sum_{m=0}^{n-1} \sum_{j=0}^m \sum_{k=0}^i (-1)^{m-j} \binom{m}{j} \binom{i}{k} d^{i-k} \left(\frac{X^m}{m!} \right) \nabla^{m-j+k}(e_j) \quad (1.13)$$

$$= \sum_{s=0}^{n-1+i} \sum_{j=\max(0, i-s)}^{\min(n-1, n-1+i-s)} \alpha(s; i, j) \frac{X^{s+j-i}}{(s+j-i)!} \nabla^s(e_j) , \quad (1.14)$$

where $\alpha(s; i, j)$ is

$$\alpha(s; i, j) := \left[\sum_{k=\max(0, s+j-(n-1))}^{\min(i, s)} (-1)^{s-k} \binom{s-k+j}{j} \binom{i}{k} \right] \in \mathbb{Z} . \quad (1.15)$$

In matrix form, if $H_s(X) = (h_{s;i,j}(X))_{i,j=0,\dots,n-1}$, then

$$(\nabla^i(c_0(\mathbf{e}, X)))_i = \left(\sum_{s=0}^{2n-2} H_s(X) G_s \right) \cdot (e_i)_i = \sum_{s=0}^{2n-2} \left(H_s(X) G_s \right) \cdot (e_i)_i \quad (1.16)$$

$$= \sum_{s=0}^{2n-2} \left(\sum_{j=0}^{n-1} h_{s;i,j}(X) g_{s;j,k} \right)_{i,k} \cdot (e_i)_i = \sum_{s=0}^{2n-2} \left(\sum_{k=0}^{n-1} \sum_{j=0}^{n-1} h_{s;i,j}(X) g_{s;j,k} e_k \right)_i \quad (1.17)$$

$$= \sum_{s=0}^{2n-2} \left(\sum_{j=0}^{n-1} h_{s;i,j}(X) \left(\sum_{k=0}^{n-1} g_{s;j,k} e_k \right) \right)_i = \sum_{s=0}^{2n-2} \left(\sum_{j=0}^{n-1} h_{s;i,j}(X) \nabla^s(e_j) \right)_i. \quad (1.18)$$

So that $\nabla^i(c_0(\mathbf{e}, X)) = \sum_{s=0}^{2n-2} \sum_{j=0}^{n-1} h_{s;i,j}(X) \nabla^s(e_j)$. This means that

$$h_{s;i,j}(X) = \alpha(s; i, j) \cdot \frac{X^{s+j-i}}{(s+j-i)!}. \quad (1.19)$$

□

Below we write the first examples of $H(X)$ for $n = 2, 3, 4, 5$.

$n = 2$:

$$H(X) = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} G_1 + \begin{pmatrix} 0 & 0 \\ -X & 0 \end{pmatrix} G_2$$

$n = 3$:

$$H(X) = \begin{pmatrix} 1 & X & \frac{X^2}{2!} \\ 0 & 1 & X \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -X & -2\frac{X^2}{2} & 0 \\ 0 & -X & -\frac{X^2}{2} \\ 0 & 0 & 2X \end{pmatrix} G_1 + \begin{pmatrix} \frac{X^2}{2} & 0 & 0 \\ 0 & -2\frac{X^2}{2} & 0 \\ 0 & -3X & \frac{X^2}{2} \end{pmatrix} G_2 + \begin{pmatrix} 0 & 0 & 0 \\ \frac{X^2}{2} & 0 & 0 \\ X & -2\frac{X^2}{2} & 0 \end{pmatrix} G_3 + \begin{pmatrix} 0 & 0 & 0 \\ \frac{X^2}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} G_4$$

$n = 4$:

$$H(X) = \begin{pmatrix} 1 & X & \frac{X^2}{2!} & \frac{X^3}{3!} \\ 0 & 1 & X & \frac{X^2}{2!} \\ 0 & 0 & 1 & X \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -X & -2\frac{X^2}{2} & -3\frac{X^3}{3!} & 0 \\ 0 & -X & -2\frac{X^2}{2} & \frac{X^3}{3!} \\ 0 & 0 & -X & 2\frac{X^2}{2} \\ 0 & 0 & 0 & 3X \end{pmatrix} G_1 + \begin{pmatrix} \frac{X^2}{2} & 3\frac{X^3}{3!} & 0 & 0 \\ 0 & \frac{X^2}{2} & -3\frac{X^3}{3!} & 0 \\ 0 & 0 & -5\frac{X^2}{2} & \frac{X^3}{3!} \\ 0 & 0 & -6X & 3\frac{X^2}{2} \end{pmatrix} G_2 +$$

$$\begin{pmatrix} -\frac{X^3}{3!} & 0 & 0 & 0 \\ 0 & 3\frac{X^3}{3!} & 0 & 0 \\ 0 & 4\frac{X^2}{2} & -3\frac{X^3}{3!} & 0 \\ 0 & 4X & -8\frac{X^2}{2} & \frac{X^3}{3!} \end{pmatrix} G_3 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{X^3}{3!} & 0 & 0 & 0 \\ -\frac{X^2}{2} & 3\frac{X^3}{3!} & 0 & 0 \\ -X & 7\frac{X^2}{2} & -3\frac{X^3}{3!} & 0 \end{pmatrix} G_4 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{X^3}{3!} & 0 & 0 & 0 \\ -2\frac{X^2}{2} & 3\frac{X^3}{3!} & 0 & 0 \end{pmatrix} G_5 + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{X^3}{3!} & 0 & 0 & 0 \end{pmatrix} G_6$$

$n = 5$:

$$H(X) = \begin{pmatrix} 1 & X & \frac{X^2}{2!} & \frac{X^3}{3!} & \frac{X^4}{4!} \\ 0 & 1 & X & \frac{X^2}{2!} & \frac{X^3}{3!} \\ 0 & 0 & 1 & X & \frac{X^2}{2} \\ 0 & 0 & 0 & 1 & X \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -X & -2\frac{X^2}{2!} & -3\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \\ 0 & -X & -2\frac{X^2}{2!} & -3\frac{X^3}{3!} & \frac{X^4}{4!} \\ 0 & 0 & -X & -2\frac{X^2}{2!} & 2\frac{X^3}{3!} \\ 0 & 0 & 0 & -X & 3\frac{X^2}{2!} \\ 0 & 0 & 0 & 0 & 4X \end{pmatrix} G_1 + \begin{pmatrix} \frac{X^2}{2!} & 3\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \\ 0 & \frac{X^2}{2!} & 3\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \\ 0 & 0 & \frac{X^2}{2!} & -7\frac{X^3}{3!} & \frac{X^4}{4!} \\ 0 & 0 & 0 & -9\frac{X^2}{2!} & 3\frac{X^3}{3!} \\ 0 & 0 & 0 & -10X & 6\frac{X^2}{2!} \end{pmatrix} G_2 +$$

$$\begin{pmatrix} -\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 & 0 & 0 \\ 0 & -\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \\ 0 & 0 & 9\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \\ 0 & 0 & 10\frac{X^2}{2!} & -11\frac{X^3}{3!} & \frac{X^4}{4!} \\ 0 & 0 & 10X & -20\frac{X^2}{2!} & 4\frac{X^3}{3!} \end{pmatrix} G_3 + \begin{pmatrix} \frac{X^4}{4!} & 0 & 0 & 0 & 0 \\ 0 & -4\frac{X^4}{4!} & 0 & 0 & 0 \\ 0 & -5\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \\ 0 & -5\frac{X^2}{2!} & 15\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \\ 0 & -5X & 25\frac{X^2}{2!} & -13\frac{X^3}{3!} & \frac{X^4}{4!} \end{pmatrix} G_4 +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{X^4}{4!} & 0 & 0 & 0 & 0 \\ \frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 & 0 & 0 \\ \frac{X^2}{2!} & -9\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \\ X & -14\frac{X^2}{2!} & 21\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 \end{pmatrix} G_5 + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{X^4}{4!} & 0 & 0 & 0 & 0 \\ 2\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 & 0 & 0 \\ 3\frac{X^2}{2!} & -13\frac{X^3}{3!} & 6\frac{X^4}{4!} & 0 & 0 \end{pmatrix} G_6 + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{X^4}{4!} & 0 & 0 & 0 & 0 \\ 3\frac{X^3}{3!} & -4\frac{X^4}{4!} & 0 & 0 & 0 \end{pmatrix} G_7 +$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{X^4}{4!} & 0 & 0 & 0 & 0 \end{pmatrix} G_8$$

2. Small connections are cyclic over an ultrametric Banach algebra.

In this section we provide a sufficient condition for differential modules over an ultrametric Banach algebras, in order to guarantee that the Katz's vector (1.2) is a cyclic vector.

2.1 Norms and matrices

We recall that an ultrametric norm on a commutative ring with unit \mathcal{B} is a map $|\cdot| : \mathcal{B} \rightarrow \mathbb{R}_{\geq 0}$ satisfying $|0| = 0$, $|1| = 1$, $|a + b| \leq \max(|a|, |b|)$, $|ab| \leq |a| \cdot |b|$, for all $a, b \in \mathcal{B}$. We require moreover $|na| = |n||a|$ for all $n \in \mathbb{Z}$, $a \in \mathcal{B}$. Hence, in particular, the norm on \mathbb{Z} induced by $|\cdot|$ is ultrametric and so $|n| \leq 1$ for all $n \in \mathbb{Z}$. If \mathcal{B} is complete and separated⁶ with respect to $|\cdot|$ then we say that \mathcal{B} is an *ultrametric Banach algebra*. A norm on $M_n(\mathcal{B})$ is a map $\|\cdot\| : M_n(\mathcal{B}) \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\|0\| = 0$, $\|1\| = 1$, $\|A + B\| \leq \max(\|A\|, \|B\|)$, $\|AB\| = \|A\| \cdot \|B\|$, $\|bA\| = |b|\|A\|$ for all $b \in \mathcal{B}$, $A, B \in M_n(\mathcal{B})$. In the sequel we will consider on $M_n(\mathcal{B})$ two norms

$$\text{sup-norm:} \quad |(a_{i,j})| \quad := \sup_{i,j} |a_{i,j}|, \quad (2.1)$$

$$\rho\text{-sup-norm:} \quad |(a_{i,j})|^{(\rho)} := \sup_{i,j} |a_{i,j}| \rho^{j-i}, \quad \rho > 0. \quad (2.2)$$

Notice that if \mathcal{C} is a \mathcal{B} -algebra together with a norm $|\cdot|_{\mathcal{C}}$ extending⁷ that of \mathcal{B} , and if $c \in \mathcal{C}$ is an element with norm $|c| = \rho^{-1}$, then $|A|^{(\rho)} = |\Lambda_c^{-1} A \Lambda_c|$, for all $A \in M_n(\mathcal{B})$, where Λ_c is the diagonal matrix with diagonal equal to $(1, c, c^2, \dots, c^{n-1})$.

2.1.1 Norm of derivation Let $(\mathcal{B}, |\cdot|)$ be an ultrametric Banach algebra, and let $\|\cdot\| : M_n(\mathcal{B}) \rightarrow \mathbb{R}_{\geq 0}$ be a fixed norm. Let now $d : \mathcal{B} \rightarrow \mathcal{B}$ be a continuous derivation. We extend d to $M_n(\mathcal{B})$ by $d((a_{i,j})_{i,j}) := (d(a_{i,j}))_{i,j}$. Let $|d|$ denotes the norm operator of d acting on \mathcal{B} :

$$|d| := \sup_{b \neq 0, b \in \mathcal{B}} \frac{|d(b)|}{|b|}. \quad (2.3)$$

We will always assume that the norm $\|\cdot\|$ verifies

$$\|d(A)\| \leq |d| \cdot \|A\| \quad (2.4)$$

for all $A \in M_n(\mathcal{B})$. This holds for the sup-norm and the ρ -sup-norm.

2.2 Norm of the matrix of the connection and cyclic vectors

We consider as above an ultrametric Banach algebra $(\mathcal{B}, |\cdot|)$, together with a continuous derivation $d : \mathcal{B} \rightarrow \mathcal{B}$. Let $\|\cdot\| : M_n(\mathcal{B}) \rightarrow \mathbb{R}_{\geq 0}$ be a fixed norm satisfying (2.4), for which $M_n(\mathcal{B})$ is complete and separated. Let (M, ∇) be a differential module. We assume that there is a element $t \in \mathcal{B}$ such that $d(t) = 1$. In order to consider the Katz's base change matrix (1.8) we assume that $(n-1)!$ is invertible in \mathcal{B} . As in the above sections we denote by G_n the matrix of the n -th iterated connection $\nabla^n : M \rightarrow M$ with respect to a basis \mathbf{e} . The simple idea of this section is the following.

LEMMA 2.1. *If the matrices G_1, \dots, G_{2n-2} are small enough in norm, in order to verify*

$$\|H_0(-t)H_s(t)G_s\| < 1 \quad (2.5)$$

for all $s = 1, \dots, 2n-2$, then the Katz's base change matrix

$$H(t) := H_0(t) + H_1(t)G_1 + \dots + H_{2n-2}(t)G_{2n-2} \quad (2.6)$$

is invertible.

⁶ \mathcal{B} is separated if and only if $|a| = 0$ implies $a = 0$.

⁷i.e. in order that the structural morphism $\mathcal{B} \rightarrow \mathcal{C}$ is an isometry

Proof. Indeed $H_0(t)$ is always invertible with inverse

$$H_0(t)^{-1} = H_0(-t). \quad (2.7)$$

So that $H(t)$ is invertible if and only if $H_0(t)^{-1}H(t) = 1 + \sum_{s=1}^{2n-2} H_0(-t)H_s(t)G_s$ is invertible. \square

Of course a sufficient condition to have (2.5) is

$$\|G_s\| < (\|H_s(t)\| \cdot \|H_0(-t)\|)^{-1}. \quad (2.8)$$

In the following subsection we provide an explicit upper bound on the sup-norm and on the ρ -sup-norm of $G := G_1$ sufficient to guarantee (2.8) for all $s = 1, \dots, 2n-2$. In order to do that we relate the norm of G_s with that of G_1 by the following

LEMMA 2.2. *For all $s \geq 1$ one has*

$$\|G_s\| \leq \|G_1\| \cdot \sup(\|G_1\|, |d|)^{s-1}. \quad (2.9)$$

Proof. We have the recursive relation $G_{s+1} = d(G_s) + G_s G_1$. Since we assume $\|d(G_s)\| \leq |d|\|G_s\|$, then one easily has $\|G_{s+1}\| \leq \|G_s\| \cdot \max(|d|, \|G_1\|)$. By induction the lemma is proved. \square

2.3 Upper bound for the sup-norm.

Let now the chosen norm $\|\cdot\| = |\cdot|$ be the sup-norm (2.1). We are looking for a condition on $|G_1|$ that guarantee

$$|G_s| < (|H_s(t)| \cdot |H_0(-t)|)^{-1}, \quad (2.10)$$

for all $s = 1, \dots, 2n-2$. Thanks to Proposition 1.6 one has⁸

$$|H_0(t)| = |H_0(-t)| = \sup_{i=0, \dots, n-1} |t^i|/|i!| \quad (2.11)$$

$$|H_s(t)| \leq |H_0(-t)|, \text{ for all } s = 1, \dots, 2n-2. \quad (2.12)$$

Indeed since $\alpha(s; i, j)$ is an integer, and since the norm $|\cdot|$ is ultrametric, one has $|\alpha(s; i, j)| \leq 1$. From this we have

$$(|H_0(-t)| |H_s(t)|)^{-1} \geq |H_0(t)|^{-2}. \quad (2.13)$$

On the other hand by Lemma 2.2 one has

$$|G_s| \leq |G_1| \cdot \max(|G_1|, |d|)^{s-1}. \quad (2.14)$$

Hence it is enough to prove that

$$|G_1| \cdot \max(|G_1|, |d|)^{s-1} < |H_0(t)|^{-2}, \quad (2.15)$$

for all $s = 1, \dots, 2n-2$.

PROPOSITION 2.3. *Assume that*

$$|G_1| < |H_0(t)|^{-2} \cdot \min\left(1, \frac{1}{|d|^{2n-3}}\right) = \min\left(1, \frac{1}{|t|}, \frac{|2|}{|t^2|}, \dots, \frac{|(n-1)!|}{|t^{n-1}|}\right)^2 \cdot \min\left(1, \frac{1}{|d|^{2n-3}}\right). \quad (2.16)$$

Then (M, ∇) is cyclic and the Katz's vector $c_0(\mathbf{e}, t)$ is a cyclic vector for M .

Proof. We observe that both minimums are ≤ 1 , moreover $\min(1, 1/|t|, |2|/|t^2|, \dots, |(n-1)!|/|t^{n-1}|) \leq 1/|t|$. Since $d(t) = 1$, then $|d||t| \geq 1$, and hence $1/|t| \leq |d|$. Our assumption then implies $|G_1| < |d|$. Hence (2.15) becomes $|G_1| \cdot |d|^{s-1} < |H_0(t)|^{-2}$ for all $s = 1, \dots, 2n-2$. This inequality is fulfilled if and only if $|G_1| < \min_{s=1, \dots, 2n-2} |H_0(t)|^{-2}/|d|^{s-1} = |H_0(t)|^{-2} \cdot \min(1, 1/|d|^{2n-3})$ which is our assumption. \square

⁸Notice that $|\cdot|$ is not assumed to be multiplicative, hence $|t^i| \leq |t|^i$.

2.4 Upper bound for the ρ -sup-norm with $\rho = |t|^{-1}$.

We assume that

$$\rho := |t|^{-1}, \quad |(a_{i,j})_{i,j}|^{(|t|^{-1})} = \sup_{i,j} |a_{i,j}| |t|^{i-j}. \quad (2.17)$$

As above we shall provide a condition on G_1 to guarantee

$$|G_s|^{(|t|^{-1})} < (|H_s(t)|^{(|t|^{-1})} \cdot |H_0(-t)|^{(|t|^{-1})})^{-1}, \quad (2.18)$$

for all $s = 1, \dots, 2n - 2$. Since $|\cdot|$ is not assumed to be multiplicative, hence $|t^i| \leq |t|^i$. This implies

$$|H_0(-t)|^{(|t|^{-1})} = |H_0(t)|^{(|t|^{-1})} = \sup_{i,j=0,\dots,n-1} \frac{|t^i| |t|^{-i}}{|i!|} \leq \sup_{i=0,\dots,n-1} \frac{1}{|i!|} = \frac{1}{|(n-1)!|}. \quad (2.19)$$

Of course if $|\cdot|$ is power multiplicative⁹, the above inequality is actually an equality. Notice that since $|\cdot|$ is ultrametric on \mathbb{Z} , then $|(n-1)!| \leq 1$.

LEMMA 2.4. *One has*

$$|H_s(t)|^{(|t|^{-1})} \leq \frac{|t^s|}{|(n-1)!|}. \quad (2.20)$$

Proof. Thanks to proposition 1.6, for all $s = 1, \dots, 2n - 2$ one has

$$|H_s(t)|^{(|t|^{-1})} = \max_{i,j=0,\dots,n-1} |h_{s;i,j}| |t|^{i-j} = \max_{i,j=0,\dots,n-1} |\alpha(s; i, j)| \frac{|t^{s+j-i}|}{|(s+j-i)!|} |t|^{i-j}. \quad (2.21)$$

Now $|\alpha(s; i, j)| \leq 1$, and it is equal to 0 for $j - i \notin [\max(1-s, 1-n), n-1-s]$. So, since $|t^{s+j-i}| \leq |t^s| |t|^{j-i}$, then we obtain $|H_s(t)|^{(|t|^{-1})} \leq \max_{j-i \in [\max(1-s, 1-n), n-1-s]} \frac{|t^{s+j-i}|}{|(s+j-i)!|} |t|^{i-j} \leq \max_{r \in [\max(1-s, 1-n), n-1-s]} \frac{|t^s|}{|(s+r)!|} = \frac{|t^s|}{|(n-1)!|}$. \square

Then one has

$$(|H_0(-t)|^{(|t|^{-1})} \cdot |H_s(t)|^{(|t|^{-1})})^{-1} \geq \frac{|(n-1)!|^2}{|t|^s}. \quad (2.22)$$

On the other hand by Lemma 2.2 one has

$$|G_s|^{(|t|^{-1})} \leq |G_1|^{(|t|^{-1})} \cdot \max(|G_1|^{(|t|^{-1})}, |d|)^{s-1}. \quad (2.23)$$

So condition (2.18) is fulfilled if

$$|G_1|^{(|t|^{-1})} \cdot \max(|G_1|^{(|t|^{-1})}, |d|)^{s-1} < \frac{|(n-1)!|^2}{|t|^s} \quad (2.24)$$

for all $s = 1, \dots, 2n - 2$.

PROPOSITION 2.5. *Assume that*

$$|G_1|^{(|t|^{-1})} < \frac{|(n-1)!|^2 |d|}{(|d||t|)^{2n-2}}. \quad (2.25)$$

Then (M, ∇) is cyclic and the Katz's vector $c_0(\mathbf{e}, t)$ is a cyclic vector for M .

Proof. Since $d(t) = 1$, then $|d||t| \geq 1$. Our assumption then implies $|G_1|^{(|t|^{-1})} < |(n-1)!|^2 |d| \leq |d|$. Hence (2.24) becomes $|G_1|^{(|t|^{-1})} \cdot |d|^{s-1} < \frac{|(n-1)!|^2}{|t|^s}$ for all $s = 1, \dots, 2n - 2$. This inequality is fulfilled if and only if $|G_1|^{(|t|^{-1})} < \min_{s=1,\dots,2n-2} \frac{|(n-1)!|^2 |d|}{(|t||d|)^s} = \frac{|(n-1)!|^2 |d|}{(|t||d|)^{2n-2}}$ which is our assumption. \square

⁹The norm $|\cdot|$ is power multiplicative if it verifies $|b^n| = |b|^n$ for all $b \in \mathcal{B}$, and all integer $n \geq 0$

2.5 Upper bound for the ρ -sup-norm with $\rho = |d|$.

We now set

$$\rho := |d|, \quad |(a_{i,j})_{i,j}|^{(|d|)} = \sup_{i,j} |a_{i,j}| |d|^{j-i}. \quad (2.26)$$

We quickly reproduce the computations of section 2.4. As usual we have to prove that $|G_s|^{(|d|)} < (|H_0(-t)|^{(|d|)} \cdot |H_s(t)|^{(|d|)})^{-1}$. One has

$$|H_0(-t)|^{(|d|)} = |H_0(t)|^{(|d|)} = \max_{i=0,\dots,n-1} \frac{|t^i| |d|^i}{|i!|} \leq \max_{i=0,\dots,n-1} \frac{(|d||t|)^i}{|i!|} \quad (2.27)$$

As usual this becomes an equality if $|\cdot|$ is power multiplicative.

LEMMA 2.6. *Let $\rho \geq 1$ be a real number, and let $s \geq 0$ be an integer. The sequence of real numbers $i \mapsto \rho^i / |(s+i)!|$ is increasing.*

Proof. One has $\rho^{i+1} / |(s+i+1)!| \geq \rho^i / |(s+i)!|$ if and only if $\rho / |s+i+1| \geq 1$. This last is true since the norm of a integer is ≤ 1 , because the norm is ultrametric. \square

Since $d(t) = 1$, then $|d||t| \geq 1$, so we then have

$$|H_0(-t)|^{(|d|)} \leq \frac{(|d||t|)^{n-1}}{|(n-1)!|}. \quad (2.28)$$

LEMMA 2.7. *One has*

$$|H_s(t)|^{(|d|)} \leq |t^s| \cdot \frac{(|d||t|)^{n-1-s}}{|(n-1)!|} \quad (2.29)$$

Proof. As in lemma 2.4 one has

$$|H_s(t)|^{(|d|)} = \max_{i,j} |\alpha(s; i, j)| \frac{|t^{s+j-i}| |d|^{j-i}}{|(s+j-i)!|} \leq \max_{i,j} \frac{|t^s| (|d||t|)^{j-i}}{|(s+j-i)!|} = |t^s| \cdot \max_r \frac{(|d||t|)^r}{|(s+r)!|}, \quad (2.30)$$

where i, j runs in $[0, n-1]$, and $r \in [\max(1-s, 1-n), n-1-s]$. By Lemma 2.6 the last maximum is equal to $(|d||t|)^{n-1-s} / |(n-1)!|$. \square

Then one has

$$(|H_0(-t)|^{(|d|)} \cdot |H_s(t)|^{(|d|)})^{-1} \geq \frac{|(n-1)!|^2}{|t^s| \cdot (|d||t|)^{2n-2-s}}. \quad (2.31)$$

As usual one also has $|G_s|^{(|d|)} \leq |G_1|^{(|d|)} \cdot \max(|G_1|^{(|d|)}, |d|)^{s-1}$, so what we need is

$$|G_1|^{(|d|)} \cdot \max(|G_1|^{(|d|)}, |d|)^{s-1} < \frac{|(n-1)!|^2}{|t|^s (|d||t|)^{2n-2-s}} \quad (2.32)$$

for all $s = 1, \dots, 2n-2$.

PROPOSITION 2.8. *Assume that*

$$|G_1|^{(|d|)} < \frac{|(n-1)!|^2 |d|}{(|d||t|)^{2n-2}}. \quad (2.33)$$

Then (M, ∇) is cyclic and the Katz's vector $c_0(\mathbf{e}, t)$ is a cyclic vector for M .

Proof. Since $d(t) = 1$, then $|d||t| \geq 1$. Our assumption then implies $|G_1|^{(|d|)} < |(n-1)!|^2 |d| \leq |d|$. Hence (2.32) becomes $|G_1|^{(|d|)} \cdot |d|^{s-1} < \frac{|(n-1)!|^2}{|t|^s (|d||t|)^{n-1-s}}$, for all $s = 1, \dots, 2n-2$. But this is actually our assumption. \square

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